Conservation Laws of Variable Coefficient Diffusion–Convection Equations

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We study local conservation laws of variable coefficient diffusion–convection equations of the form $f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u)u_x$. The main tool of our investigation is the notion of equivalence of conservation laws with respect to the equivalence groups. That is why, for the class under consideration we first construct the usual equivalence group $G^\sim$ and the extended one $\hat{G}^\sim$ including transformations which are nonlocal with respect to arbitrary elements. The extended equivalence group $\hat{G}^\sim$ has interesting structure since it contains a non-trivial subgroup of gauge equivalence transformations. Then, using the most direct method, we carry out two classifications of local conservation laws up to equivalence relations generated by $G^\sim$ and $\hat{G}^\sim$, respectively. Equivalence with respect to $\hat{G}^\sim$ plays the major role for simple and clear formulation of the final results.

1 Introduction

In this paper we study local conservation laws of PDEs of the general form

$$f(x)u_t = (g(x)A(u)u_x)_x + h(x)B(u)u_x, \tag{1}$$

where $f = f(x)$, $g = g(x)$, $h = h(x)$, $A = A(u)$ and $B = B(u)$ are arbitrary smooth functions of their variables, and $f(x)g(x)A(u)\neq 0$.

Conservation laws were investigated for some subclasses of class (1). In particular, Dorodnitsyn and Svirshchevskii [2] (see also [4, Chapter 10]) constructed the local conservation laws for the class of reaction–diffusion equations of the form $u_t = (A(u)u_x)_x + C(u)$, which has non-empty intersection with the class under consideration. The first-order local conservation laws of equations (1) with $f = g = h = 1$ were constructed by Kara and Mahomed [6]. Developing the results obtained in [1] for the case $hB = 0$, $f = 1$, in the recent papers [5, 11] we completely classified potential conservation laws (including arbitrary order local ones) of equations (1) with $f = g = h = 1$ with respect to the corresponding equivalence group.

For class (1) in Section 2 we first construct the usual equivalence group $G^\sim$ and the extended one $\hat{G}^\sim$ including transformations which are nonlocal with respect to arbitrary elements. We discuss the structure of the extended equivalence group $\hat{G}^\sim$ having non-trivial subgroup of gauge equivalence transformations. Then we carry out two classifications of local conservation laws up to the equivalence relations generated by $G^\sim$ and $\hat{G}^\sim$, respectively, using the most direct method (Section 3).

The main tool of our investigation is the notion of equivalence of conservation laws with respect to equivalence groups, which was introduced in [11]. Below we adduce some necessary notions and statements, restricting ourselves to the case of two independent variables. See [9, 11] for more details and general formulations.

Let $\mathcal{L}$ be a system $L(t,x,u_{(\rho)}) = 0$ of PDEs for unknown functions $u = (u^1,\dots,u^m)$ of independent variables t (the time variable) and x (the space variable). Here $u_{(\rho)}$ denotes the set of all the partial derivatives of the functions u of order no greater than ρ, including u as the derivatives of the zero order.

Definition 1. A *conservation law* of the system $\mathcal{L}$ is a divergence expression

$$D_t F(t,x,u_{(r)}) + D_x G(t,x,u_{(r)}) = 0 \tag{2}$$

which vanishes for all solutions of $\mathcal{L}$. Here D_t and D_x are the operators of total differentiation with respect to t and x, respectively; F and G are correspondingly called the *density* and the *flux* of the conservation law.

Two conserved vectors (F,G) and (F',G') are *equivalent* if there exist functions $\hat{F}$, $\hat{G}$ and H of t, x and derivatives of u such that $\hat{F}$ and $\hat{G}$ vanish for all solutions of $\mathcal{L}$ and $F' = F+\hat{F}+D_xH$, $G' = G+\hat{G}-D_tH$.

Lemma 1. *[11] Any point transformation g between systems $\mathcal{L}$ and $\tilde{\mathcal{L}}$ induces a linear one-to-one mapping g_* between the corresponding linear spaces of conservation laws.*

Consider the class $\mathcal{L}|_S$ of systems $L(t,x,u_{(\rho)},\theta(t,x,u_{(\rho)})) = 0$ parameterized with the parameter-functions $\theta = \theta(t,x,u_{(\rho)})$. Here L is a tuple of fixed functions of t, x, $u_{(\rho)}$ and θ. θ denotes the tuple of arbitrary (parametric) functions $\theta(t,x,u_{(\rho)}) = (\theta^1(t,x,u_{(\rho)}),\dots,\theta^k(t,x,u_{(\rho)}))$ satisfying the additional condition $S(t,x,u_{(\rho)},\theta_{(q)}(t,x,u_{(\rho)})) = 0$.

Let $P = P(L,S)$ denote the set of pairs each from which consists of a system from $\mathcal{L}|_S$ and a conservation law of this system. Action of transformations from an equivalence group $G^{\sim}$ of the class $\mathcal{L}|_S$ together with the pure equivalence relation of conserved vectors naturally generates an equivalence relation on P. Classification of conservation laws with respect to $G^{\sim}$ will be understood as classification in P with respect to the above equivalence relation. This problem can be investigated in the way that it is similar to group classification in classes of systems of differential equations. Specifically, we firstly construct the conservation laws that are defined for all values of the arbitrary elements. (The corresponding conserved vectors may depend on the arbitrary elements.) Then we classify, with respect to the equivalence group, arbitrary elements for each of the systems that admits additional conservation laws.

2 Equivalence transformations and choice of investigated class

In order to classify the conservation laws of equations of the class (1), firstly we have to investigate equivalence transformations of this class.

The usual equivalence group $G^{\sim}$ of class (1) is formed by the nondegenerate point transformations in the space of (t,x,u,f,g,h,A,B), which are projectible on the space of (t,x,u), i.e. they have the form

$$\begin{aligned}&(\tilde{t},\tilde{x},\tilde{u}) = (T^t,T^x,T^u)(t,x,u),\\ &(\tilde{f},\tilde{g},\tilde{h},\tilde{A},\tilde{B}) = (T^f,T^g,T^h,T^A,T^B)(t,x,u,f,g,h,A,B),\end{aligned} \tag{3}$$

and transform any equation from the class (1) for the function $u = u(t,x)$ with the arbitrary elements (f,g,h,A,B) to an equation from the same class for function $\tilde{u} = \tilde{u}(\tilde{t},\tilde{x})$ with the new arbitrary elements $(\tilde{f},\tilde{g},\tilde{h},\tilde{A},\tilde{B})$.

Theorem 1. *$G^{\sim}$ consists of the transformations*

$$\tilde t=\delta_1 t+\delta_2,\quad \tilde x=X(x),\quad \tilde u=\delta_3 u+\delta_4,$$
$$\tilde f=\frac{\varepsilon_1\delta_1 f}{X_x(x)},\quad \tilde g=\varepsilon_1\varepsilon_2^{-1}X_x(x)g,\quad \tilde h=\varepsilon_1\varepsilon_3^{-1}h,\quad \tilde A=\varepsilon_2 A,\quad \tilde B=\varepsilon_3 B,$$

where δ_j $(j=\overline{1,4})$ and ε_i $(i=\overline{1,3})$ are arbitrary constants, $\delta_1\delta_3\varepsilon_1\varepsilon_2\varepsilon_3\neq 0$, X is an arbitrary smooth function of x, $X_x\neq 0$.

It appears that class (1) admits other equivalence transformations which do not belong to $G^{\sim}$ and form, together with usual equivalence transformations, an extended equivalence group. We demand for these transformations to be point with respect to (t,x,u). The explicit form of the new arbitrary elements $(\tilde f,\tilde g,\tilde h,\tilde A,\tilde B)$ is determined via (t,x,u,f,g,h,A,B) in some non-fixed (possibly, nonlocal) way. We construct the complete (in this sense) extended equivalence group $\hat G^{\sim}$ of class (1), using the direct method.

Existence of such transformations can be explained in many respects by features of representation of equations in the form (1). This form leads to an ambiguity since the same equation has an infinite series of different representations. More exactly, two representations (1) with the arbitrary element tuples (f,g,h,A,B) and $(\tilde f,\tilde g,\tilde h,\tilde A,\tilde B)$ determine the same equation iff

$$\tilde f=\varepsilon_1\varphi f,\quad \tilde g=\varepsilon_1\varepsilon_2^{-1}\varphi g,\quad \tilde h=\varepsilon_1\varepsilon_3^{-1}\varphi h,\quad \tilde A=\varepsilon_2 A,\quad \tilde B=\varepsilon_3(B+\varepsilon_4 A),\tag{4}$$

where $\varphi=\exp\left(-\varepsilon_4\int\frac{h(x)}{g(x)}dx\right)$, ε_i $(i=\overline{1,4})$ are arbitrary constants, $\varepsilon_1\varepsilon_2\varepsilon_3\neq 0$ (the variables t, x and u do not transform!).

The transformations (4) act only on arbitrary elements and do not really change equations. In general, transformations of such type can be considered as trivial [7] ("gauge") equivalence transformations and form the "gauge" (normal) subgroup $\hat G^{\sim g}$ of the extended equivalence group $\hat G^{\sim}$. Application of "gauge" equivalence transformations is equivalent to rewriting equations in another form. In spite of really equivalence transformations, their role in group classification comes not as a choice of representatives in equivalence classes but as a choice of the form of these representatives.

Let us note that transformations (4) with $\varepsilon_4\neq 0$ are nonlocal with respect to arbitrary elements, otherwise they belong to $G^{\sim}$ and form the "gauge" (normal) subgroup $G^{\sim g}$ of the equivalence group $G^{\sim}$.

The factor-group $\hat G^{\sim}/\hat G^{\sim g}$ coincides for class (1) with $G^{\sim}/G^{\sim g}$ and can be assumed to consist of the transformations

$$\tilde t=\delta_1 t+\delta_2,\quad \tilde x=X(x),\quad \tilde u=\delta_3 u+\delta_4,$$
$$\tilde f=\frac{\delta_1 f}{X_x(x)},\quad \tilde g=X_x(x)g,\quad \tilde h=h,\quad \tilde A=A,\quad \tilde B=B,\tag{5}$$

where δ_i $(i=\overline{1,4})$ are arbitrary constants, $\delta_1\delta_3\neq 0$, X is an arbitrary smooth function of x, $X_x\neq 0$.

Using the transformation $\tilde t=t$, $\tilde x=\int\frac{dx}{g(x)}$, $\tilde u=u$ from $G^{\sim}/G^{\sim g}$, we can reduce equation (1) to $\tilde f(\tilde x)\tilde u_{\tilde t}=(A(\tilde u)\tilde u_{\tilde x})_{\tilde x}+\tilde h(\tilde x)B(\tilde u)\tilde u_{\tilde x}$, where $\tilde f(\tilde x)=g(x)f(x)$, $\tilde g(\tilde x)=1$ and $\tilde h(\tilde x)=h(x)$. (Likewise any equation of form (1) can be reduced to the same form with $\tilde f(\tilde x)=1$.) That is why, without loss of generality we restrict ourselves to investigation of the equation

$$f(x)u_t=(A(u)u_x)_x+h(x)B(u)u_x.\tag{6}$$

Any transformation from $\hat{G}^{\sim}$, which preserves the condition $g = 1$, has the form

$$\begin{aligned}&\tilde{t} = \delta_1 t + \delta_2, \quad \tilde{x} = \delta_5 \int e^{\delta_8 \int h\, dx} dx + \delta_6, \quad \tilde{u} = \delta_3 u + \delta_4, \quad \tilde{f} = \delta_1 \delta_5^{-1} \delta_9 f e^{-2\delta_8 \int h\, dx},\\ &\tilde{h} = \delta_9 \delta_7^{-1} h e^{-\delta_8 \int h\, dx}, \quad \tilde{A} = \delta_5 \delta_9 A, \quad \tilde{B} = \delta_7 (B + \delta_8 A),\end{aligned} \tag{7}$$

where δ_i $(i = \overline{1,9})$ are arbitrary constants, $\delta_1\delta_3\delta_5\delta_7\delta_9 \neq 0$. The set $\hat{G}_1^{\sim}$ of such transformations is a subgroup of $\hat{G}^{\sim}$. It can be considered as a generalized equivalence group of class (6) after admitting dependence of (3) on arbitrary elements [8] and additional supposition that such dependence can be nonlocal. The group $G_1^{\sim}$ of usual (local) equivalence transformations of class (6) coincides with the subgroup singled out from $\hat{G}_1^{\sim}$ via the condition $\delta_8 = 0$. The transformations (7) with non-vanishing values of the parameter δ_8 are nonlocal and are compositions of (nonlocal) gauge and usual equivalence transformations from $G_1^{\sim}$.

There exists a way to avoid operations with nonlocal in (t, x, u) equivalence transformations. More exactly, we can assumed that the parameter-function B is determined up to an additive term proportional to A and subtract such term from B before applying equivalence transformations (5).

3 Local conservation laws

We search local conservation laws of equations from class (6).

Lemma 2. *Any conservation law of form* (2) *of any equation from class* (6) *is equivalent to a conservation law that has the density depending on* t, x, *and* u *and the flux depending on* t, x, u *and* u_x.

Note 1. A similar statement is true for an arbitrary (1+1)-dimensional evolution equation $\mathcal{L}$ of the even order $r = 2\bar{r}$, $\bar{r} \in \mathbb{N}$. For example [3], for any conservation law of $\mathcal{L}$ we can assume up to equivalence of conserved vectors that F and G depend only on t, x and derivatives of u with respect to x, and the maximal order of derivatives in F is not greater than $\bar{r}$.

Theorem 2. *A complete list of* $G_1^{\sim}$*-inequivalent equations* (6) *having nontrivial conservation laws is exhausted by the following ones*

1. $h = 1: \quad (\, fu,\ -Au_x - \smallint B\,).$
2. $h = x^{-1}: \quad (\, xfu,\ -xAu_x + \smallint A - \smallint B\,).$
3. $B = \varepsilon A: \quad (yfe^{-\varepsilon \smallint h}u,\ -yAe^{-\varepsilon \smallint h}u_y + \smallint A),\ (fe^{-\varepsilon \smallint h}u,\ -Ae^{-\varepsilon \smallint h}u_y).$
4. $B = \varepsilon A + 1, \quad f = -hZ^{-1}, \quad h = Z^{-1/2} \exp\left(-\int \dfrac{a_{00} + a_{11}}{2Z} dy\right):$

   $(\,(\sigma^{k1}y + \sigma^{k0})fe^{-\varepsilon \smallint h}u,\ -(\sigma^{k1}y + \sigma^{k0})(Ae^{-\varepsilon \smallint h}u_y + hu) + \sigma^{k1} \smallint A\,)$
5. $B = \varepsilon A + 1, \quad f = h_y: \quad (\, e^{t-\varepsilon \smallint h}h_y u,\ -e^t(Ae^{-\varepsilon \smallint h}u_y + hu)\,).$
6. $B = \varepsilon A + 1, \quad f = h_y + hy^{-1}: \quad (\, e^{t-\varepsilon \smallint h}yfu,\ -e^t(yAe^{-\varepsilon \smallint h}u_y + yhu - \smallint A)\,).$
7. $A = 1, \quad B_u \neq 0, \quad f = -h(h^{-1})_{xx}: \quad (\, e^t(h^{-1})_{xx}u,\ e^t(h^{-1}u_x - (h^{-1})_x u + \displaystyle\int B)\,).$
8. $A = 1, \quad B = 0: \quad (\, \alpha fu,\ -\alpha u_x + \alpha_x u\,).$

Here y is implicitly determined by the formula $x = \int e^{\varepsilon \int h(y)dy} dy$; $\varepsilon, a_{ij} = \mathrm{const}$, $i, j = \overline{0,1}$; $(\sigma^{k1}, \sigma^{k0}) = (\sigma^{k1}(t), \sigma^{k0}(t))$, $k = \overline{1,2}$, is a fundamental solution of the system of ODEs $\sigma^{\nu}_t = a_{\mu\nu}\sigma^{\mu}$; $Z = a_{01}y^2 + (a_{00} - a_{11})y - a_{10}$; $\alpha = \alpha(t,x)$ is an arbitrary solution of the linear equation $f\alpha_t + \alpha_{xx} = 0$. (Together with constraints on A, B, f and h we also adduce complete lists of linear independent conserved vectors.)

In Theorem 2 we classify conservation laws with respect to the usual equivalence group $G_1^{\sim}$. The results that are obtained can be formulated in an implicit form only, and indeed Case 4 is split into a number of inequivalent cases depending on values of a_{ij}. At the same time, using the extended equivalence group $\hat{G}_1^{\sim}$, we can present the result of classification in a closed and simple form with a smaller number of inequivalent equations having nontrivial conservation laws.

Theorem 3. *A complete list of $\hat{G}_1^{\sim}$-inequivalent equations* (6) *having nontrivial conservation laws is exhausted by the following ones*

1. $h = 1: \quad (\,fu,\ -Au_x - \textstyle\int B\,)$.

2a. $B = 0: \quad (\,fu,\ -Au_x\,),\ (\,xfu,\ -xAu_x + \textstyle\int A\,)$.

2b. $B = 1, \quad f = 1, \quad h = 1: \quad (\,u,\ -Au_x - u\,)$,
$(\,(x+t)u,\ -(x+t)(Au_x + u) + \textstyle\int A\,)$.

2c. $B = 1, \quad f = e^x, \quad h = e^x: \quad (\,e^{x+t}u,\ -e^t(Au_x + e^x u)\,)$,
$(\,e^{x+t}(x+t)u,\ -e^t(x+t)(Au_x + e^x u) + e^t \textstyle\int A\,)$.

2d. $B = 1, \quad f = x^{\mu-1}, \quad h = x^{\mu}: \quad (\,x^{\mu-1}e^{\mu t}u,\ -e^{\mu t}(Au_x + x^{\mu}u)\,)$,
$(\,x^{\mu}e^{(\mu+1)t}u,\ e^{(\mu+1)t}(-xAu_x - x^{\mu+1}u + \textstyle\int A)\,)$.

3. $B = 1, \quad f = e^{\mu/x}x^{-3}, \quad h = e^{\mu/x}x^{-1}, \quad \mu \in \{0,1\}:$
$(\,fe^{-\mu t}xu,\ -e^{-\mu t}x(Au_x + hu) + e^{-\mu t}\textstyle\int A\,)$,
$(\,fe^{-\mu t}(tx-1)u,\ -e^{-\mu t}(tx-1)(Au_x + hu) + te^{-\mu t}\textstyle\int A\,)$.

4. $B = 1,\ f = |x-1|^{\mu-3/2}|x+1|^{-\mu-3/2},\ h = |x-1|^{\mu-1/2}|x+1|^{-\mu-1/2}:$
$(\,fe^{(2\mu+1)t}(x-1)u,\ -e^{(2\mu+1)t}(x-1)(Au_x + hu) + e^{(2\mu+1)t}\textstyle\int A\,)$,
$(\,fe^{(2\mu-1)t}(x+1)u,\ -e^{(2\mu-1)t}(x+1)(Au_x + hu) + e^{(2\mu-1)t}\textstyle\int A\,)$.

5. $B = 1, \quad f = e^{\mu\arctan x}(x^2+1)^{-3/2}, \quad h = e^{\mu\arctan x}(x^2+1)^{-1/2}:$
$(\,fe^{\mu t}(x\cos t + \sin t)u,\ -e^{\mu t}(x\cos t + \sin t)(Au_x + hu) + e^{\mu t}\cos t\textstyle\int A\,)$,
$(\,fe^{\mu t}(x\sin t - \cos t)u,\ -e^{\mu t}(x\sin t - \cos t)(Au_x + hu) + e^{\mu t}\sin t\textstyle\int A\,)$.

6. $B = 1, \quad f = h_x: \quad (\,e^t h_x u,\ -e^t(Au_x + hu)\,)$.

7. $B = 1, \quad f = h_x + hx^{-1}: \quad (\,e^t xfu,\ -e^t(xAu_x + xhu - \textstyle\int A)\,)$.

8. $A = 1, \quad B_u \neq 0, \quad f = -h(h^{-1})_{xx}: \quad (\,e^t(h^{-1})_{xx}u,\ e^t(h^{-1}u_x - (h^{-1})_x u + \textstyle\int B)\,)$.

9. $A = 1, \quad B = 0: \quad (\,\alpha fu,\ -\alpha u_x + \alpha_x u\,)$.

Here $\mu = \mathrm{const}$, $\alpha = \alpha(t,x)$ is an arbitrary solution of the linear equation $f\alpha_t + \alpha_{xx} = 0$. (Together with constraints on A, B, f and h we also adduce complete lists of linear independent conserved vectors.)

Note 2. The cases 2b–2d can be reduced to the case 2a by means of additional equivalence transformations:

2b→2a: $\tilde{t} = t$, $\tilde{x} = x + t$, $\tilde{u} = u$;

2c→2a: $\tilde{t} = e^t$, $\tilde{x} = x + t$, $\tilde{u} = u$;

2d $(\mu + 1 \neq 0)$→2a: $\tilde{t} = (\mu + 1)^{-1}(e^{(\mu+1)t} - 1)$, $\tilde{x} = e^t x$, $\tilde{u} = u$;

2d $(\mu + 1 = 0)$→2a: $\tilde{t} = t$, $\tilde{x} = e^t x$, $\tilde{u} = u$.

4 Conclusion

The present paper is the beginning for further studies on this subject. For the class under consideration we intend to perform a complete classification of potential conservation laws and construct an exhaustive list of locally inequivalent potential systems corresponding to them. These results can be developed and generalized in a number of different directions. So, studying different kinds of symmetries (Lie, nonclassical, generalized ones) of constructed potential systems, we may obtain the corresponding kinds of potential symmetries (usual potential, nonclassical potential, generalized potential). Analogously, local equivalence transformations between potential systems constructed for different initial equations result in nonlocal (potential) equivalence transformations for the class under consideration (see e.g. [10]). In such way it is possible to find new nonlocal connections between variable coefficient diffusion–convection equations. We believe that the same approach used in this article, can be employed for investigation of wider classes of differential equations.

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